A Note on the Fowler's Solution of the Lane-Emden Equation of Index 3

In this note, the validity of the FOWLER's solution is extended to a larger region around the centre. A series solution is also sought in the (ξ, θ) plane, which covers the FOWLER's solution near the origin[1].

The Lane-Emden differential equation for $n = 3$ is

$$y \frac{dy}{dz} + y + z^3 = 0, \quad (1)$$

where

$$y = \frac{dz}{dt}, \quad z = \xi\theta = \theta e^{-t}.$$

Applying the HARDY's[2] theorem that $dy/dz \to 0$ as $t \to \infty$ or $\xi \to 0$ for the algebraic differential equation (1), FOWLER[3] obtained the approximate solution $\theta \sim 1/\xi$ $(2 \log C/\xi)^{-1/2}$ near the origin, where C is a constant of integration. Now we seek a solution valid in a larger region $-1 < dy/dz < 1$ around the origin $\xi = 0$. The equation (1) can be written in the form

$$-\frac{y}{z^3} = 1 - \frac{dy}{dz} + \left(\frac{dy}{dz}\right)^2 - \left(\frac{dy}{dz}\right)^3 + \dots (-1)^n \left(\frac{dy}{dz}\right)^n + \dots \quad (2)$$

We choose a sufficiently large positive integer m such that for $n \geq m$, $(dy/dz)^n \to 0$ in the region $-1 < dy/dz < 1$. If we substitute for dy/dz from (1) in (2) and sum up the terms, we get

$$(y + z^3)^m = 0 \quad \text{or} \quad y = \frac{dz}{dt} = -z^3, \quad (3)$$

which, on integration, yields a solution in the (ξ, θ) plane in the form

$$\theta = \frac{1}{\xi} \left[\frac{1}{2 \log C/\xi}\right]^{1/2}, \quad (4)$$

where C is a constant of integration. The solution (4) includes the FOWLER's solution near the centre and thus extends the validity of the FOWLER's approximation to a larger region around the centre.

The homology theorem shows that we can take $z = 1$ at $t = 0$ without any loss of generality. Therefore, at $t = 0$, we can assume a Taylor's expansion of the form

$$z = 1 + z_0^{(1)} t + \frac{1}{2!} z_0^{(2)} t^2 + \frac{1}{3!} z_0^{(3)} t^3 + \frac{1}{4!} z_0^{(4)} t^4 + \dots \quad (5)$$

where $z_0^{(n)}$ stands for the value of nth derivative of z at $t = 0$. Calculating different derivatives with the help of (3) and substituting in (5), we get

$$z = 1 - t + \frac{3}{2} t^2 - \frac{5}{2} t^3 + \frac{35}{8} t^4 - \frac{63}{8} t^5 + \dots$$

or reverting to the (ξ, θ) plane we obtain

$$\theta = \frac{1}{\xi} \left[1 + \log\xi + \frac{3}{2} (\log\xi)^2 + \frac{5}{2} (\log\xi)^3 + \frac{35}{8} (\log\xi)^4 + \dots\right]. \quad (6)$$

The series solution given by (6) is in full agreement with the FOWLER's approximate solution near the origin and with the solution given by (4) towards the centre as well as towards the boundary.

Zusammenfassung. Für FOWLER's Lösung der Lane-Emden-Gleichung von Index 3 wird der Gültigkeitsbereich erweitert.

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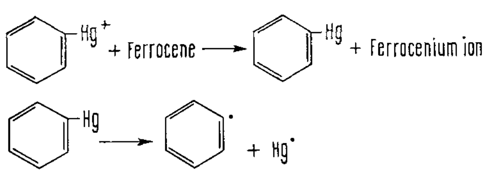
Reduction of Benzene Tetrakis-Mercuric Acetate with Ferrocene

It is well known that solvolysis of alkylmercuric salts leads to the formation of carbonium ions[1] while arylmercuric salts resist such heterolysis[2]. In the presence of reducing agents such as ferrocene, however, phenylmercuric acetate afforded phenyl radical[3] via the formation of phenylmercury (I) intermediate.



Similar reactions were observed in the solvolysis of triethyllead acetate and subsequent reduction of the triethyllead cation to form triethyllead which further fragments to diethyllead and ethyl radical[4]. In this paper we wish to report that solvolysis and reduction of benzene tetrakismercuric acetate have led to the formation of benzyne type intermediate.

When the synthesis of phenylene-1, 2-dimercuric acetate was attempted by reacting mercuric acetate and benzene under various conditions[5], it always resulted in the formation of a mixture of unidentifiable products. However, when mercuric acetate (Merck, Reagent grade)

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